JOURNAL OF DIFFERENTIAL EQUATIONS 15, 106–128 (1974)

Linear Autonomous Neutral Functional Differential Equations

DANIEL HENRY

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506

Received January 11, 1973

1. INTRODUCTION

Functional differential equations (FDEs) of neutral type are generalizations of ordinary differential equations and difference equations, including for example

$$x'(t) = f(t, x(t), x'(t-1), x(t-1))$$

and

$$x'(t) = g\left(t, x(t), \int_{-1}^{0} \gamma(\theta)\, x'(t+\theta)\, d\theta\right).$$

This class includes retarded FDEs, but is more general in that the derivative may appear with a time lag. We will discuss linear autonomous neutral FDEs such as $x'(t) = Ax'(t-1) + Bx(t) + Cx(t-1)$, and the main results are natural generalizations of facts familiar for ordinary differential equations and retarded FDEs [1]. Suppose the roots of the characteristic equation (2.5) are split by the line in the complex plane $\operatorname{Re}\lambda = \alpha$ into two sets, each bounded away from the line. There is a corresponding exponential dichotomy of the solutions of the homogeneous equation (2.4), the solutions in one subspace being $O(e^{\alpha_1 t})$ as $t \to +\infty$ $(\alpha_1 < \alpha)$, while in a complementary subspace the solutions exist for all $t \leqslant 0$ and are $O(e^{\alpha_2 t})$ as $t \to -\infty$ $(\alpha_2 > \alpha)$.

If $\{T(t), t \geqslant 0\}$ is the solution semigroup, the operators which map the initial state into the state at time $t \geqslant 0$, the above exponential dichotomy is equivalent to having the spectrum $\sigma(T(t))$ disjoint from the circle of radius $e^{\alpha t}$ about the origin for some $t > 0$. The point spectrum and residual spectrum of $T(t)$ are known from the general theory of semigroups [2], but the continuous spectrum is not so easily determined. We will first discuss a simplified problem, a difference equation, and obtain the exponential dichotomy by Laplace transform methods. This gives us the location of the spectrum for the difference equation. The general neutral FDE has a semigroup which equals the semigroup for a difference equation, plus a compact operator; and addition of this compact operator does not change the continuous spectrum. Thus we locate $\sigma(T(t))$, and so obtain the dichotomy in the general case.

These arguments, though framed in the space of continuous functions, easily extend to other function spaces, such as the Sobolev spaces $W_p^{(1)}(1 \leqslant p \leqslant \infty)$. We sketch the corresponding results for the useful, although somewhat atypical, case of the Lipschitz continuous functions, $W_\infty^{(1)}$.

The results obtained in this way are formally almost identical with those known for retarded FDEs, and many theorems generalize to the neutral case with only minor changes in the proof. As examples, we mention stability in the first approximation, and the saddle point property [3], already extended to neutral FDEs (with some extra hypotheses) by Hale [4]. By use of the space of Lipschitz continuous functions, several troublesome perturbation problems (state-dependent lags, asymptotically constant lags) fit naturally into the familiar mold of stability in the first approximation. Other applications, ripe for generalization, may be found in [5].

This paper, in its several incarnations, has benefitted from the comments of Jack K. Hale, and bears a heavy debt to the work of Hale and Mayer [6], Hale [7], and Hale and Cruz [8]. Thanks, Jack.

2. The Initial Value Problem

Let E^n denote the Euclidean space of real or complex column n-vectors, R the real line, and let r be a fixed positive number. $C = C([-r, 0], E^n)$ is the space of continuous functions $\phi\colon [-r, 0] \to E^n$ with norm $\|\phi\| = \sup\{|\phi(\theta)| : -r \leqslant \theta \leqslant 0\}$. If x is a continuous map of $[a - r, b)$ into E^n, then $x_t \in C$ is given, for each $a \leqslant t < b$, by

$$x_t(\theta) = x(t + \theta), \qquad -r \leqslant \theta \leqslant 0.$$

Let D, L be fixed continuous linear functionals from C into E^n. The initial value problem is: given $s \in R$, $\phi \in C$ and continuous $H\colon [s, \infty) \to E^n$, find continuous $x\colon [s - r, \infty) \to E^n$ such that $x_s = \phi$ and for $t \geqslant s$

$$(d/dt)\{D(x_t) - H(t)\} = L(x_t). \tag{2.1}$$

The functionals D, L may be represented by Stieltjes integrals

$$D(\phi) = \phi(0) - \int_{-r}^{0} d\mu(\theta)\,\phi(\theta), \qquad L(\phi) = \int_{-r}^{0} d\eta(\theta)\,\phi(\theta)$$

for $\phi \in C$, where μ, η are matrix valued functions of bounded variation which vanish at $\theta = 0$ and are left-continuous in $(-r, 0)$. We assume:

(i) μ is continuous at 0, so

$$\int_{-\epsilon}^{0} | d\mu(\theta)| \to 0 \qquad \text{as} \quad \epsilon \to 0+;$$

(ii) μ has no singular part, i.e.

$$\int_{-r}^{0} d\mu(\theta)\, \phi(\theta) = \sum_{k=1}^{\infty} A_k \phi(-w_k) + \int_{-r}^{0} A(\theta)\, \phi(\theta)\, d\theta,$$

where

$$0 < w_k \leqslant r \qquad \text{and} \qquad \sum_{k=1}^{\infty} | A_k | + \int_{-r}^{0} | A(\theta)|\, d\theta < \infty.$$

The first assumption implies [6] that the initial value problem is well posed; the second is used in Lemmas 3.5 and 4.1. It is not clear whether (ii) is necessary, but a theorem of Wiener and Pitt [9], generalizing the result [10] used in Lemma 3.5, suggests the necessity of some such restriction.

If $x = x(s, \phi, H)$ is the solution of the initial value problem (2.1) such that $x_s = \phi$, we may write

$$x_t(s, \phi, H) = T(t - s)\phi + K(t, s)H, \qquad t \geqslant s, \tag{2.2}$$

thus defining strongly continuous families of bounded linear operators $T(t)\colon C \to C$ for $t \geqslant 0$, and $K(t, s)\colon C([s, t], E^n) \to C$ for $t \geqslant s$. It should be noted that $K(t, s)H$ actually depends only on the restriction $H|_{[s,t]}$, but this minor abuse of notation should not cause confusion. By uniqueness we have:

$$\begin{gathered} T(0) = 1, \qquad T(t)\, T(s) = T(t + s) \quad \text{for } t, \quad s \geqslant 0; \\ K(s, s) = 0, \qquad T(t - s)\, K(s, \sigma)H = K(t, \sigma)H - K(t, s)H \quad \text{for } t \geqslant s \geqslant \sigma. \end{gathered} \tag{2.3}$$

Thus $\{T(t)\}_{t\geqslant 0}$ is a strongly continuous semigroup of operators on C, and this semigroup will be our main object of study. The infinitesimal generator A is defined by

$$A\phi = \lim_{t\to 0+}(1/t)[T(t)\phi - \phi],$$

the domain $\mathscr{D}(A)$ consisting of those $\phi \in C$ for which this limit exists in the topology of C. Hale and Meyer [6] proved the following.

THEOREM 2.1. *For the linear autonomous* FDE

$$(d/dt)\, D(x_t) = L(x_t) \tag{2.4}$$

where D, L *are as in* (2.1), *if* $\{T(t)\}_{t\geqslant 0}$ *is the solution semigroup with infinitesimal generator* A, *then*:

(i) *The domain* $\mathscr{D}(A) = \{\phi \in C \mid \phi' \in C$ *and* $D(\phi') = L(\phi)\}$, *and for* $\phi \in \mathscr{D}(A)$, $A\phi(\theta) = \phi'(\theta)$, $-r \leqslant \theta \leqslant 0$.

(ii) $\mathscr{D}(A)$ *is a dense subset of* C, $T(t)\,\mathscr{D}(A) \subset \mathscr{D}(A)$, *and for* $\phi \in \mathscr{D}(A)$

$$(d/dt)\, T(t)\phi = T(t)\, A\phi = AT(t)\phi.$$

(iii) *The spectrum of* A *is all point spectrum (eigenvalues)*, $\sigma(A) = P\sigma(A)$, *and consists of all complex* λ *satisfying the characteristic equation*

$$\det \Delta(\lambda) = 0 \tag{2.5}$$

where

$$\begin{aligned}\Delta(\lambda) &= \lambda D(e^{\lambda\cdot}) - L(e^{\lambda\cdot}) \\ &= \lambda I - \lambda \int_{-r}^{0} e^{\lambda\theta}\, d\mu(\theta) - \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta).\end{aligned}$$

If $\lambda \notin \sigma(A)$ *and* $\phi \in C$ *then*

$$(\lambda - A)^{-1}\, \phi(\theta) = be^{\lambda\theta} + \int_{\theta}^{0} e^{\lambda(\theta-\xi)}\, \phi(\xi)\, d\xi,$$

where

$$b = \Delta(\lambda)^{-1} \left\{D(\phi) + \int_{-r}^{0} (\lambda\, d\mu(\theta) + d\eta(\theta)) \int_{\theta}^{0} e^{\lambda(\theta-\xi)}\, \phi(\xi)\, d\xi\right\}.$$

(iv) *If* μ *is a zero of* $\det \Delta(\cdot)$ *of multiplicity* m, *then we have* $C = \mathscr{N}(\mu - A)^m \oplus \mathscr{R}(\mu - A)^m$, *the direct sum of subspaces invariant under* $T(t)$, *with dimension* $\mathscr{N}(\mu - A)^m$ *finite (in fact, dimension* $=m$ [11]). *If* $k \geqslant m$ *then* $\mathscr{N}(\mu - A)^k = \mathscr{N}(\mu - A)^m$, $\mathscr{R}(\mu - A)^k = \mathscr{R}(\mu - A)^m$, *and we write* $\mathscr{M}_\mu(A) = \mathscr{N}(\mu - A)^m$: *the generalized eigenspace corresponding to* μ. *The projection* E_μ *onto* $\mathscr{M}_\mu(A)$, *along* $\mathscr{R}(\mu - A)^m$, *is*

$$E_\mu = (1/2\pi i) \int_{|\lambda-\mu|=\delta} (\lambda - A)^{-1}\, d\lambda = \operatorname{Res}_{\lambda=\mu}(\lambda - A)^{-1}.$$

The above decomposition may clearly be made for any finite set of eigenvalues (see [6]); in the theory below, we allow infinite sets of eigenvalues, but the finite case is most important in applications.

Corresponding facts are known for the adjoint equation [12]. One method of calculating the above decomposition, i.e., calculating E_μ, uses the adjoint equation (see [6, 12]).

3. The Difference Equation

Consider the difference equation

$$x(t) = \sum_{k=1}^{\infty} A_k x(t - w_k), \qquad t \geqslant 0,$$

where $0 < w_k \leqslant r$, $\sum | A_k | < \infty$, and $\sum_{w_k \leqslant \epsilon} | A_k | \to 0$ as $\epsilon \to 0+$. This equation may be rewritten

$$D^0 x_t = 0, \qquad t \geqslant 0, \tag{3.1}$$

where D^0 is the difference operator

$$D^0 \phi = \phi(0) - \sum_{k=1}^{\infty} A_k \phi(-w_k) \qquad \text{for} \quad \phi \in C.$$

Clearly the initial value x_0 for (3.1) must be in the null space $\mathscr{N}(D^0)$, so we restrict attention to $C \cap \mathscr{N}(D^0) =_{df} C^0$. If $x(\phi)$ is the solution of (3.1) with initial value $\phi \in C^0$, we may write

$$x_t(\phi) = T^0(t)\phi, \qquad t \geqslant 0, \tag{3.2}$$

thus defining a strongly continuous semigroup of operators $\{T^0(t)\}_{t \geqslant 0}$ on C^0. The infinitesimal generator A^0 is readily shown to be (cf. [6])

$$\mathscr{D}(A^0) = \{\phi \in C^0 \mid \phi' \in C^0\}, \qquad A^0 \phi = \phi' \quad \text{for} \quad \phi \in C^0,$$

and $\sigma(A^0) = P\sigma(A^0) = \{\lambda \mid h(\lambda) = 0\}$ where

$$h(\lambda) = \det H(\lambda), \qquad H(\lambda) = I - \sum_{k=1}^{\infty} A_k e^{-\lambda w_k}. \tag{3.3}$$

Also, for $\lambda \notin \sigma(A^0)$,

$$(\lambda - A^0)^{-1} \phi(\theta) = b e^{\lambda\theta} + \int_\theta^0 e^{\lambda(\theta - \xi)} \phi(\xi)\, d\xi$$

where

$$b = H(\lambda)^{-1} \sum_{k=1}^{\infty} A_k \int_{-w_k}^{0} e^{-\lambda(w_k + \xi)} \phi(\xi)\, d\xi.$$

Since $h(\lambda)$ is an analytic almost periodic function of λ, we have for real $\alpha < \beta$,

LEMMA 3.1. (See [13, p. 268] for proof.) (i) *There exists a number N such that, for all real t, there are no more than N zeros of h in the box*

$$\{\lambda \mid \alpha \leqslant \operatorname{Re} \lambda \leqslant \beta,\, t \leqslant \operatorname{Im} \lambda \leqslant t+1\}.$$

(ii) *For any $\delta > 0$ there exists $m(\delta) > 0$ such that, whenever $\alpha \leqslant \operatorname{Re} \lambda \leqslant \beta$ and λ is at a distance $\geqslant \delta$ from every zero of h, one has $|h(\lambda)| \geqslant m(\delta)$.*

LEMMA 3.2. *If $h(\lambda_1) = 0$, then there exist $\lambda_2, \lambda_3, \ldots$ such that $h(\lambda_k) = 0$ and as $k \to \infty$, $|\lambda_k| \to \infty$ and $\operatorname{Re} \lambda_k \to \operatorname{Re} \lambda_1$.*

Proof. Suppose not; then by Lemma 3.1(ii) $h(\lambda)$ is bounded from zero for $\operatorname{Re} \lambda = \operatorname{Re} \lambda_1$, $|\lambda|$ large, which contradicts the almost periodicity of h.

LEMMA 3.3. *If $\alpha < \beta$ and $h(\lambda)$ is bounded from zero on $\operatorname{Re} \lambda = \alpha$ and $\operatorname{Re} \lambda = \beta$, then there exist rectangular contours $C_1, C_2, \ldots$*

$$C_j : \operatorname{Re} \lambda = \alpha \text{ or } \beta, \quad |\operatorname{Im} \lambda| \leqslant l_j; \qquad \alpha < \operatorname{Re} \lambda < \beta, \quad \operatorname{Im} \lambda = \pm l_j$$

with $j \leqslant l_j < j+1$ $(j = 1, 2, \ldots)$ and such that $h(\lambda)$ is uniformly bounded from zero on these contours.

Proof. It suffices to find C_j uniformly bounded away from the zeros of h (Lemma 3.1(ii)); and by Lemma 3.1(i) we may choose $l_j \in [j, j+1)$ so the distance from $\operatorname{Im} \lambda = \pm l_j$ to any zero of h in $\alpha \leqslant \operatorname{Re} \lambda \leqslant \beta$ is at least $1/4N$, independent of j.

The next result is a special case of [12, Theor. 1].

LEMMA 3.4. *Suppose $y(\cdot)$ is continuous and $D^0 y_t = 0$ for $t \geqslant 0$; then $y(t) = -\sum_{k=1}^{\infty} \int_{-r}^{0-} d_\beta X(t - \beta - w_k) A_k y(\beta)$ for $t \geqslant 0$ where $X(\cdot)$ is the $n \times n$ matrix valued function defined by*

$$X(t) = 0 \qquad \textit{for} \quad t < 0,$$

$$X(t) = I + \sum_{k=1}^{\infty} X(t - w_k) A_k \qquad \textit{for} \quad t \geqslant 0,$$

and

$$X(t) = X(t+) \textit{ at jumps.}$$

LEMMA 3.5. *Suppose α is real and nonzero, and $h(\lambda)$ is bounded away from* *zero on* $\operatorname{Re}\lambda = \alpha$. *Then for* $t \geqslant 0$, $X(t) = X^P(t) + X^Q(t)$ *where* $X^{P,Q}$ *are right continuous and* (*except for countably many* t)

$$X^P(t) = (1/2\pi i) \lim_{j\to\infty} \int_{C_j} e^{\lambda t}\lambda^{-1}H^{-1}(\lambda)\, d\lambda$$

$$X^Q(t) = (1/2\pi i) \lim_{j\to\infty} \int_{\alpha-il_j}^{\alpha+il_j} e^{\lambda t}\lambda^{-1}H^{-1}(\lambda)\, d\lambda.$$

Here $\{C_j\}_{j=1}^{\infty}$ *is the sequence of contours given by Lemma* 3.3, *with* β *chosen sufficiently large that* $\beta > \alpha$, $\beta > 0$ *and* $\beta > \sup\{\operatorname{Re}\lambda \mid h(\lambda) = 0\}$.

For $t < 0$, *define* $X^P(t)$ *by the contour integral above; then* $D^0X_t^P = 1$ (*if* $\alpha < 0$) *or* $D^0X_t^P = 0$ (*if* $\alpha > 0$) *for* $-\infty < t < \infty$. *Finally, there exists a constant* M *such that*

$$\operatorname{Var} X_t^Q = \operatorname{Var}_{[t-r,t]} X^Q \leqslant Me^{\alpha t} \qquad (t \geqslant 0),$$

$$\operatorname{Var} X_t^P \leqslant Me^{\alpha t} \qquad (t \leqslant 0).$$

Proof. Define X^P, X^Q by the above integrals, and observe $X(t) = X^P(t) + X^Q(t)$ $(t > 0)$ since the Laplace transform of $X(\cdot)$ is $\lambda^{-1}H(\lambda)^{-1}$. Also note that, for $t < 0$,

$$X^P(t) = -\frac{1}{2\pi i} \lim_{j\to\infty} \int_{\alpha-il_j}^{\alpha+il_j} e^{\lambda t}\lambda^{-1}H(\lambda)^{-1}\, d\lambda,$$

since $\int_{\beta-i\infty}^{\beta+i\infty} e^{\lambda t}\, \lambda^{-1}H(\lambda)^{-1}\, d\lambda = 0$ for $t < 0$ and the integrals over the segments $\operatorname{Im}\lambda = \pm l_j$ are $O(1/l_j)$ by Lemma 3.3.

Since $h(\lambda)$ is bounded away from zero on $\operatorname{Re}\lambda = \alpha$, it follows by a result of Cameron and Pitt [10] that

$$1/h(\lambda) = \sum_{k=1}^{\infty} b_k e^{\lambda\gamma_k}$$

with the γ_k real and $\sum_{k=1}^{\infty} | b_k | e^{\alpha\gamma_k} < \infty$. It follows that $H(\lambda)^{-1} = \sum_{k=1}^{\infty} X_k e^{\lambda\gamma_k}$ with $\sum | X_k | e^{\alpha\gamma_k} < \infty$, hence for $t > 0$, $t + \gamma_k \neq 0$,

$$\begin{aligned} X^Q(t) &= \sum_{k=1}^{\infty} X_k (1/2\pi i) \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda(t+\gamma_k)}\lambda^{-1}\, d\lambda \\ &= \sum_{\alpha(t+\gamma_k)>0} X_k \qquad \text{(cf. [14, p. 430]).} \end{aligned}$$

Therefore

$$\operatorname{Var} X_t^Q \leqslant \sum_{0\leqslant t+\gamma_k \leqslant r} | X_k | \leqslant \left(\sum | X_k | e^{\gamma_k\alpha} e^{|\alpha| r}\right) e^{\alpha t}, \qquad t > 0.$$

$\operatorname{Var} X_t^P$ is estimated similarly for $t < 0$.

THEOREM 3.1. *Suppose* $h(\lambda)$ *is bounded away from zero for* $\operatorname{Re}\lambda = \alpha$; *then* $C^0 = C \cap \mathcal{N}(D^0) = P \oplus Q$, *the direct sum of closed subspaces invariant under* $T^0(t)$ $(t \geqslant 0)$, *and*

$$P = \overline{\operatorname{span}}\{\mathcal{M}_\lambda(A^0) \mid h(\lambda) = 0, \operatorname{Re}\lambda > \alpha\}.$$

The semigroup $\{T^0(t)\mid_P, t \geqslant 0\}$ *may be extended uniquely as a group* $(-\infty < t < \infty)$ *of operators on P. There exist constants* M' *and* $\delta > 0$ *such that*

$$\| T^0(t)\phi^Q \| \leqslant M' e^{(\alpha-\delta)t} \| \phi^Q \| \qquad \textit{for} \quad t \geqslant 0, \quad \phi^Q \in Q,$$

$$\| T^0(t)\phi^P \| \leqslant M' e^{(\alpha+\delta)t} \| \phi^P \| \qquad \textit{for} \quad t \leqslant 0, \quad \phi^P \in P.$$

Proof. First observe that it suffices to consider $\delta = 0$ and $\alpha \neq 0$, since the hypothesis on α remains true for $\alpha \pm \delta$, δ small. Now let $\phi \in C^0$ and define $y = y(\phi)$ by $y_t(\phi) = T^0(t)\phi$ for $t \geqslant 0$. We may write $y(t) = y^P(t) + y^Q(t)$, where $y^{P,Q}(t) = -\sum_{k=1}^{\infty} \int_{-r}^{0-} d_\beta X^{P,Q}(t - \beta - w_k)\, A_k \phi(\beta)$, with $y^P(t)$ defined for $-\infty < t < \infty$. By Lemma 3.5,

$$| y^Q(t)| \leqslant \left(\sum | A_k |\right) M e^{\alpha t} | \phi | \qquad \text{for} \quad t \geqslant 0$$

and

$$| y^P(t)| \leqslant \left(\sum | A_k |\right) M e^{\alpha t} | \phi | \qquad \text{for} \quad t \leqslant 0.$$

Using the integral for $X^P(t)$ and the fact $\sum | A_k | < \infty$, it is easy to see (for smooth ϕ)

$$y^P(t) = \lim_{j\to\infty} \frac{1}{2\pi i} \int_{C_j} e^{\lambda t} H^{-1}(\lambda) \sum_{k=1}^{\infty} A_k \int_{-w_k}^{0} e^{-\lambda(w_k+\xi)} \phi(\xi)\, d\xi\, d\lambda$$

so

$$y^P(\theta) = \lim_{j\to\infty} \frac{1}{2\pi i} \int_{C_j} (\lambda - A^0)^{-1} \phi(\theta)\, d\lambda, \qquad -r \leqslant \theta \leqslant 0.$$

Define $E\phi(\theta) = y^P(\theta)$, $-r \leqslant \theta \leqslant 0$; then E is a continuous projection of C^0 onto $\overline{\operatorname{span}}\{\mathcal{M}_\lambda(A^0) \mid \operatorname{Re}\lambda > \alpha\}$, and we may define $P = \mathcal{R}(E)$, $Q = \mathcal{N}(E)$, so $C^0 = P \oplus Q$. Since $T^0(t)(\lambda - A^0)^{-1} = (\lambda - A^0)^{-1} T^0(t)$, it follows that E commutes with $T^0(t)$, so P, Q are invariant. Finally, $y_t^P = T^0(t) E\phi$ for $t \geqslant 0$, and $D^0 y_t^P = 0$ for $-\infty < t < \infty$, and backward continuation on the whole line is unique [15]. Therefore $y_t^P(\phi)$ defines the unique extension of $T^0(t)\mid_P$ on $-\infty < t < \infty$ such that $T^0(t)\mid_P T^0(s)\mid_P = T^0(t+s)\mid_P$ for all s, t.

THEOREM 3.2. *Let $Z = \{\operatorname{Re}\lambda \mid h(\lambda) = 0\}$ be nonempty and let $\bar{Z}$ = closure of Z; then for $t > 0$,*

$$\overline{\{e^{\lambda t} \mid h(\lambda) = 0\}} \subseteq \sigma(T^0(t)) \subseteq \{\mu \mid |\mu| = e^{\zeta t}, \zeta \in \bar{Z}\}.$$

If $\alpha \notin \bar{Z}$ and $C^0 = P \oplus Q$ is the decomposition given by Theorem 3.1 then

$$\sigma(T^0(t)\,|_P) \subseteq \{\mu \mid |\mu| = e^{\zeta t}, \zeta \in \bar{Z} \text{ and } \zeta > \alpha\}$$

$$\sigma(T^0(t)\,|_Q) \subseteq \{\mu \mid |\mu| = e^{\zeta t}, \zeta \in \bar{Z} \text{ and } \zeta < \alpha\}.$$

If Z is empty, then $\sigma(T^0(t)) = \{0\}$ for $t > 0$; in fact, $T^0(t) = 0$ for $t \geqslant rn$.

Proof. If Z is empty, then in Theorem 3.1 we may pick arbitrary α, $C^0 = P \oplus Q$ with $P = \{0\}$ and $T^0(mt)\,|_Q = T^0(mt) = O(e^{\alpha mt})$ as $m \to +\infty$ (m = integer), so the spectral radius $r(T^0(t)) = \lim_{m\to\infty} \| T^0(mt)\|^{1/m} \leqslant e^{\alpha t}$. But α is arbitrary so $r(T^0(t)) = 0$. In fact, $T^0(t) = 0$ for $t \geqslant rn$, according to [15, Theorem 1].

Now suppose Z is nonempty, choose $\alpha \notin \bar{Z}$ and decompose $C^0 = P \oplus Q$ as in Theorem 3.1. By the same argument as above, for $t > 0$,

$$r(T^0(t)\,|_Q) \leqslant \lim_{m\to\infty} [M' e^{(\alpha-\delta)mt}]^{1/m} = e^{(\alpha-\delta)t}$$

and $r(T^0(-t)\,|_P) \leqslant e^{-(\alpha+\delta)t}$, so

$$\sigma(T^0(t)\,|_P) = \{\mu \mid 1/\mu \in \sigma(T^0(-t)\,|_P)\} \subset \{\mu \mid |\mu| \geqslant e^{(\alpha+\delta)t}\}.$$

Consider the class of all closed subsets K of the real line such that for $t > 0$,

$$\sigma(T^0(t)) \subset \{\mu \mid |\mu| = e^{kt}, k \in K\}.$$

Clearly every such set contains $\bar{Z}$, so if K_m is the smallest such set (i.e., $K_m = \bigcap K$) then $K_m \supset \bar{Z}$. In fact, $K_m = \bar{Z}$; for if there exists $\alpha \in K_m \setminus \bar{Z}$, then we may decompose as above and find $\delta > 0$ such that $K_m \setminus (\alpha - \delta, \alpha + \delta)$ is a smaller set of the class considered.

EXAMPLE. Consider the scalar difference operator $D^0\phi = \phi(0) - \phi(-1)$, with $h(\lambda) = 1 - e^{-\lambda}$ and $\sigma(A^0) = \{2\pi i n \mid n = \text{integer}\}$. For $t > 0$, closure $\{e^{2\pi i n t} \mid n = 0, \pm 1, \pm 2, \ldots\} \subset \sigma(T^0(t)) \subset$ unit circle, with equality when t is irrational, but not when t is rational. In fact, for $t = 1, 2, 3, \ldots$, $T^0(t) =$ identity and $\sigma(T^0(t)) = \{1\}$.

Calculation of $\bar{Z}$. The problem of finding the zeros of h is important, but little studied. The many papers "on the zeros of exponential sums" generally allow polynomial terms which are dominant for large values of the argument, thus simplifying the expression. Some references and results are in [14].

There are two extreme cases where the situation is fairly simple, for $h(\lambda) = \sum_{k=0}^{N} a_k e^{-\lambda w_k}$, the $a_k \neq 0$, $0 = w_0 < w_1 < \cdots < w_N$.

(i) The w_k are commensurable.

Suppose the w_k are integral multiples of some $\beta > 0$; then $h(\lambda)$ is a polynomial in $e^{-\beta\lambda}$ say $h(\lambda) = a_N \prod_{\nu=1}^{M} (e^{-\lambda\beta} - r_\nu)$, and

$$Z = \bar{Z} = \{-(1/\beta) \log | r_\nu | \mid \nu = 1,..., M\}.$$

(ii) The w_k are rationally independent.

Suppose $\sum_{k=1}^{N} m_k w_k = 0$ (m_k = integer) only if all $m_k = 0$; then $\bar{Z}$ is a finite union of closed intervals, with $\rho \in \bar{Z}$ if and only if the $(N+1)$ lengths $| a_0 |, | a_1 | e^{-\rho w_1},..., | a_N | e^{-\rho w_N}$ can form a closed polygon, i.e., no one of these terms is greater than the sum of the other terms. In particular, the smallest closed interval containing $\bar{Z}$ is $[\rho_N, \rho_0]$ where

$$| a_k | e^{-\rho_k w_k} = \sum_{j \neq k} | a_j | e^{-\rho_k w_j} \qquad (k = 0, N).$$

To prove this, observe that $h(\rho + i\tau) = 0$ for some real τ only if $a_0 + \sum_{k=1}^{N} a_k e^{-\rho w_k} e^{i\theta_k} = 0$ for some real $\theta_1,..., \theta_N$. Conversely, if such real θ_k exist (i.e., if these lengths can form a closed polygon) then by Kronecker's theorem [16, Theor. 6.12] there exist real τ making $\{e^{-iw_k\tau} - e^{i\theta_k} \mid k = 1,..., N\}$ arbitrarily small, thus making $h(\rho + i\tau)$ arbitrarily small. This implies $\rho \in \bar{Z}$, by Lemma 3.1(ii).

In the general case, we still have $\bar{Z} \subseteq [\rho_N, \rho_0]$, but this may not be the smallest such interval. If $\{\zeta_1,..., \zeta_\nu\}$ is a rationally independent basis for $w_1,..., w_N$, then $h(\lambda)$ may be written $h(\lambda) = \sum_\alpha a_\alpha e^{-\alpha\cdot\zeta\lambda}$ where $\alpha = (\alpha_1,..., \alpha_\nu)$ has integer components and $\alpha \cdot \zeta = \sum_{k=1}^{\nu} \alpha_k \zeta_k$. If $h(\rho, \theta) = \sum_\alpha a_\alpha e^{-\alpha\cdot\zeta\rho} e^{i\alpha\cdot\theta}$, $\theta = (\theta_1,..., \theta_\nu)$, then $\rho \in \bar{Z}$ if and only if

$$h(\rho, \theta) = 0 \qquad \text{for some} \quad \theta \in [0, 2\pi] \times \cdots \times [0, 2\pi].$$

EXAMPLES. $Z = \{\mathrm{Re}\, \lambda \mid h(\lambda) = 0\}$

(i) $h(\lambda) = 1 + e^{-\lambda} + e^{-2\lambda}$, $Z = \{0\} \subset [\rho_2, \rho_0] = [-\rho_0, \rho_0]$, $\rho_0 = \log(1 + 5^{1/2})/2$.

(ii) $h(\lambda) = 1 + e^{-\lambda} + e^{-\pi\lambda}$, $\bar{Z} = [\rho_2, \rho_0]$, approximately $[-.27, .37]$.

(iii) $h(\lambda) = 1 + e^{-\lambda} + e^{-2\lambda} + e^{-\pi\lambda}$, $\bar{Z} = [\rho_3, \sigma]$, approximately $[-.56, .30]$ where $e^{-\pi\sigma} = (3)^{1/2}/2(1 - e^{-2\sigma})$, while ρ_0 is about .60.

Observe sup, inf Z may be discontinuous as functions of the w_k, but ρ_0, ρ_N are continuous.

Theorem 3.2, together with Lemma 3.2 and the remarks above give a rough picture of $\sigma(T^0(t))$ as a collection of annuli about the origin, each densely packed with eigenvalues (aside, perhaps, from special values of t). In the next section we consider far more general neutral FDEs, but the only change in this picture of the spectrum is a sprinkling of "normal" eigenvalues outside the annuli.

4. Decomposition and Estimates for the Neutral FDE

We consider first the homogeneous FDE

$$(d/dt)\, D(x_t) = L(x_t) \tag{4.1}$$

and determine $\sigma(T(t))$. It is known [2, 16.7] that the point spectrum $P\sigma(T(t)) = \{e^{\lambda t} \mid \lambda \in P\sigma(A)\}$, plus possibly zero; and the residual spectrum of $T(t)$ is empty, since A has no residual spectrum. That leaves the continuous spectrum, but now the general theory is silent, aside from a cautionary example [2, 23.16] of a strongly continuous semigroup having continuous spectrum, while its infinitesimal generator has no spectrum. For retarded equations, this is not a problem: $T(r)$ is compact, so the continuous spectrum, if any, is $\{0\}$. In the general case we show first that $T(t)$ and $T^0(t)$ differ by a compact operator, and then that addition of this compact operator does not change the continuous spectrum. The first claim is a modified form of a result of Hale [7, Lemma 5.2].

Lemma 4.1. *Suppose D, L are given as in* (2.1), *and D^0 as in* (3.1), *and let $\{T(t),\ t \geqslant 0\}$, $\{T^0(t),\ t \geqslant 0\}$ be the corresponding semigroups on C and $C \cap \mathcal{N}(D^0)$ respectively. If E^0 = projection of C onto $\mathcal{N}(D^0)$, then for $t \geqslant 0$,*

$$T(t) - T^0(t)\, E^0 \colon C \to C \textit{ is compact.}$$

Proof. If $\phi \in C$, $x_t = T(t)\phi$ for $t \geqslant 0$ and $\phi = \phi^0 + \phi^1$ where $E^0\phi = \phi^0$, then

$$\begin{aligned} D^0(x_t - \phi^1) &= \int_{-r}^{0} A(\theta)\, x(t+\theta)\, d\theta - \int_{-r}^{0} A(\theta)\, \phi(\theta)\, d\theta \\ &\quad + \int_0^t ds \int_{-r}^{0} d\eta(\theta)\, x(s+\theta) \\ D^0(x_0 - \phi^1) &= 0 \end{aligned}$$

hence

$$x_t = T(t)\phi = (I - E^0)\phi + T^0(t)\,\phi^0 + K^0(t, 0)\,H(\cdot, \phi)$$

where

$$H(t, \phi) = \int_{-r}^{0} A(\theta)(T(t)\phi)(\theta)\,d\theta + \int_0^t ds \int_{-r}^{0} d\eta(\theta)(T(s)\phi)(\theta).$$

Now $\mathscr{N}(D^0) = \mathscr{R}(E^0)$ has finite codimension, so $\mathscr{R}(I - E^0)$ has finite dimension, and $(I - E^0)$ is compact; and $K^0(t, 0)$ is continuous. It suffices, therefore, to prove $\{H(\cdot, \phi) \mid \|\phi\| \leqslant 1\}$ is contained in a compact set of $C([0, t_0], E^n)$ for each $t_0 > 0$. Since $\| T(t)\phi \| \leqslant Me^{kt}$ for some constants M, k we have for $0 \leqslant t,\ t + \delta \leqslant t_0$ and $\| \phi \| \leqslant 1$,

$$\begin{aligned}&| H(t + \delta, \phi) - H(t, \phi)| \\ &\qquad \leqslant Me^{kt_0} \left[\int_{-r-|\delta|}^{|\delta|} | A(\theta - \delta) - A(\theta)|\, d\theta + | \delta | \operatorname{Var} \eta\right]\end{aligned}$$

where $A(\theta) = 0$ for $\theta \notin [-r, 0]$. This proves equicontinuity, and boundedness is obvious, so the compactness is proved.

We also need the following result on perturbation of spectra. An eigenvalue μ of a linear operator U is a "normal eigenvalue" if it is an isolated point of $\sigma(U)$ and the corresponding generalized eigenspace $\mathscr{M}_\mu(U)$ is finite dimensional, i.e. the projection $E_\mu = 1/2\pi i \oint(\lambda - U)^{-1}\, d\lambda$ (integrated over $| \lambda - \mu | = \delta > 0$, δ small) has finite dimensional range. A "normal point" of U is a normal eigenvalue or a point of the resolvent set, and $\tilde{\rho}(U)$ denotes the (open) set of normal points of U. A slight change in the proof of Lemma 5.2 of [17, p. 22] gives

LEMMA 4.2. *Let U, V be linear operators on a Banach space with V compact, and let G be an open connected component of the set of normal points of U, $\tilde{\rho}(U)$. Then either G is a component of $\tilde{\rho}(U + V)$, or else $G \cap \rho(U) \subset P\sigma(U + V)$.*

THEOREM 4.1. *If $H(\lambda) = 1 - \sum_{k=1}^{\infty} A_k e^{-\lambda w_k}$, $Z = \{\operatorname{Re} \lambda \mid \det H(\lambda) = 0\}$ and $\Delta(\lambda) = \lambda H(\lambda) - \lambda \int_{-r}^{0} A(\theta)\, e^{\lambda\theta}\, d\theta - \int_{-r}^{0} d\eta(\theta)\, e^{\lambda\theta}$, then for $t > 0$,*

$$P\sigma(T(t))\backslash\{0\} = \{e^{\lambda t} \mid \det \Delta(\lambda) = 0\},$$

$R\sigma(T(t))$ is empty,

$$C\sigma(T(t))\backslash\{0\} \subset \{\mu \mid | \mu | = e^{\zeta t},\ \zeta \in \bar{Z}\},$$

and

$$\sigma(T(t)) \subset \{0\} \cup \{e^{\lambda t} \mid \det \Delta(\lambda) = 0\} \cup \{\mu \mid | \mu | = e^{\zeta t},\ \zeta \in \bar{Z}\}.$$

If Z is nonempty and bounded, then 0 is a normal point of $T(t)$.

Proof. The first two claims follow from [2, 16.7] and Theorem 2.1. Choose $\alpha < \beta$ such that (α, β) is disjoint from $\bar{Z}$, and let

$$I_t = \{\mu \mid e^{\alpha t} < |\mu| < e^{\beta t}\};$$

we will show I_t is disjoint from $C\sigma(T(t))$. By Theorem 3.2, I_t is in the resolvent set of $T^0(t)\,E^0$, hence by Lemma 4.1 and 4.2, either $I_t \subset \tilde{\rho}(T(t))$ or else $I_t \subset P\sigma(T(t))$. But $P\sigma(T(t))$ is countable, since $\sigma(A)$ is countable, hence I_t consists of normal points of $T(t)$ and is disjoint from $C\sigma(T(t))$.

Remark. As $|\lambda| \to \infty$ with $\operatorname{Re}\lambda$ bounded, we see $\det \Delta(\lambda) = \lambda^n h(\lambda) + o(|\lambda|^n)$, so by Rouche's theorem and Lemma 3.1, the zeros of $\det \Delta(\cdot)$ in a strip $\alpha \leqslant \operatorname{Re}\lambda \leqslant \beta$ are asymptotic to those of $h(\cdot)$ (see [6, Chap. III, Lemma]). Thus the examples and remarks following Theorem 3.2 are relevant also in the general case.

Remark. Z is bounded if $H(\lambda) = I - \sum_{k=1}^{N} A_k e^{-\lambda w_k}$ has a finite number of terms. For a necessary and sufficient condition, see [16, Theors. 3.27 and 3.29].

We use the result, proved in [12, Sect. 6].

LEMMA 4.3. *For* $t \geqslant rn$,

$$\overline{\mathscr{R}(T(t))} = \overline{\operatorname{span}}\{\mathscr{M}_\lambda(A) \mid \lambda \in \sigma(A)\}.$$

For some types of FDEs, the solution may be expressed in a convergent series of functions from the $\mathscr{M}_\lambda(A)$ (see, e.g., [14]), and this implies the above lemma. But in the general case, the adjoint theory of [12] seems to be needed.

THEOREM 4.2. *Suppose* $\alpha \notin \overline{\operatorname{Re}\sigma(A)}$, *i.e.*, $\det \Delta(\lambda) \neq 0$ *in some strip* $|\operatorname{Re}\lambda - \alpha| < \epsilon$, $\epsilon > 0$. *Then* $C = P \oplus Q$ *where* P, Q *are closed subspaces invariant under* $T(t)$,

$$P = \overline{\operatorname{span}}\{\mathscr{M}_\lambda(A) \mid \operatorname{Re}\lambda > \alpha\},$$

$$Q = \{\phi \mid E_\lambda\phi = 0 \text{ for all } \operatorname{Re}\lambda > \alpha\}$$

where E_λ = *projection onto* $\mathscr{M}_\lambda(A)$ *along* $\mathscr{R}(\lambda - A)^l$ = *residue of the resolvent of* A *at* λ.

The projection E *onto* P *along* Q *may be written*

$$E\phi = \lim_{j\to\infty} \frac{1}{2\pi i}\int_{C_j} (\lambda - A)^{-1}\phi\, d\lambda = \lim_{j\to\infty} \sum_{\lambda \text{ in } C_j} E_\lambda\phi$$

for a dense set of ϕ *in* C, *where* $\{C_j\}_{j=1}^{\infty}$ *is an expanding sequence of contours (as in*

Lemma 3.3), *bounded away from* $\sigma(A)$ *and such that each* $\lambda \in \sigma(A)$ *with* $\operatorname{Re} \lambda > \alpha$ *is inside* C_j *for sufficiently large* j.

The restriction of the semigroup to P *may be extended to a group* $\{T(t)|_p, -\infty < t < \infty\}$ *of bounded operators on* P, *with* $T(t)|_p T(s)|_p = T(t+s)|_p$ *for all* t, s.

Finally, there exist constants M_1, M_2 *and* $\delta > 0$ *such that*

$$\| T(t)\phi^Q \| \leqslant M_1 e^{(\alpha-\delta)t} \| \phi^Q \| \qquad \text{for} \quad t \geqslant 0, \phi^Q \in Q,$$

$$\| T(t)\phi^P \| \leqslant M_2 e^{(\alpha+\delta)t} \| \phi^P \| \qquad \text{for} \quad t \leqslant 0, \phi^P \in P.$$

Proof. First observe that, by Theorem 4.1, $\sigma(T(t)) = \sigma_P \cup \sigma_Q$ where σ_Q is contained in the disc $\{\mu \mid |\mu| \leqslant e^{(\alpha-\delta)t}\}$, and $\sigma_P \subset \{\mu \mid |\mu| \geqslant e^{(\alpha+\delta)t}\}$, so σ_P, σ_Q are disjoint spectral sets for each fixed $t > 0$. Therefore [19] we have $C = P \oplus Q$ with P, Q invariant under $T(t)$, $\sigma(T(t)|_P) = \sigma_P$, $\sigma(T(t)|_Q) = \sigma_Q$, and $E = 1/2\pi i \int_\Gamma (\mu - T(t))^{-1} d\mu$ is the projection onto P along Q, if Γ is a contour enclosing σ_P and excluding σ_Q. This decomposition is actually independent of $t > 0$—this follows from the semigroup property, but we prove it instead by determining P, Q as claimed in the statement of the theorem. If $\phi \in \mathscr{M}_\lambda(A)$, then $(e^{\lambda t} - T(t))^m \phi = 0$ for some m, so

$$(\mu - T(t))^{-1} \phi = \sum_{p=0}^{m-1} (\mu - e^{\lambda t})^{-p-1} (e^{\lambda t} - T(t))^p \phi,$$

hence $E\phi = \phi$ or 0, according to whether $e^{\lambda t}$ is in σ_P or σ_Q. Thus we have

$$\overline{\operatorname{span}}\{\mathscr{M}_\lambda(A) \mid \operatorname{Re} \lambda > \alpha\} \subseteq P = \mathscr{R}(E),$$

$$\overline{\operatorname{span}}\{\mathscr{M}_\lambda(A) \mid \operatorname{Re} \lambda < \alpha\} \subseteq Q = \mathscr{N}(E).$$

Now $0 \notin \sigma_P$, so $T(t)|_P$ maps P onto itself, as do $T(kt)|_P$, $k = 1, 2, \ldots$. This says $P \subseteq \mathscr{R}(T(kt)) \subseteq \overline{\operatorname{span}}\{\mathscr{M}_\lambda(A) \mid \lambda \in \sigma(A)\}$ if $kt \geqslant rn$, which proves $P = \overline{\operatorname{span}}\{\mathscr{M}_\lambda(A) \mid \operatorname{Re} \lambda > \alpha\}$.

Now E_λ commutes with E, so if $\phi \in Q$ then $E_\lambda \phi \in Q \cap \mathscr{M}_\lambda(A)$, thus $E_\lambda \phi = 0$ for $\operatorname{Re} \lambda > \alpha$. Conversely if $E_\lambda \phi = 0$ for $\operatorname{Re} \lambda > \alpha$, then $E_\lambda E\phi = 0$ for all λ, therefore $(\lambda - A)^{-1} E\phi$ is an entire function of λ (see Theor. 2.1(iii)). Arguing as in [15, Theor. 1], $(\lambda - A)^{-1} E\phi$ is of finite ($\leqslant rn$) exponential type, hence $T(t) E\phi = 0$ for $t \geqslant rn$. But $T(t)|_P$ is one-one, therefore $E\phi = 0$, $\phi \in Q$, as claimed.

In view of the first remark following Theorem 4.1 and Lemma 3.3, we may choose an expanding sequence of rectangular contours C_j (as in that lemma) which are uniformly bounded away from $\sigma(A)$, but eventually enclose every eigenvalue λ with $\operatorname{Re} \lambda > \alpha$.

Therefore

$$E\phi = \lim_{j\to\infty} \frac{1}{2\pi i} \int_{C_j} (\lambda - A)^{-1}\phi \, d\lambda$$

for $\phi \in Q$ (when both sides vanish), and for $\phi \in \mathscr{M}_\lambda(A)$, $\operatorname{Re}\lambda > \alpha$ (when both sides $= \phi$), and for $\phi \in Q \oplus \operatorname{span}\{\mathscr{M}_\lambda(A) \mid \operatorname{Re}\lambda > \alpha\}$, a dense subset of C.

For $t < 0$, define $T(t)\,|_P = [T(-t)\,|_P]^{-1}$, so $T(t)\,|_P\, T(s)\,|_P = T(t+s)\,|_P$ for $-\infty < s, t < \infty$.

The estimates follow by a standard argument [1]. For example, we prove $\| T(t)\,|_Q \| = o(e^{\alpha t})$ as $t \to \infty$; this suffices, since $\alpha \pm \delta$ also satisfy the hypothesis (for sufficiently small δ). If we assume instead that there exist $t_\nu \to \infty$ and $H > 0$ such that $\| T(t_\nu)\,|_Q \| \, e^{-\alpha t_\nu} \geqslant H$ for all ν, then the spectral radius $r(T(1)\,|_Q) \geqslant e^\alpha$, since (with $t_\nu = m_\nu + \tau_\nu$, $m_\nu =$ integer, $0 \leqslant \tau_\nu < 1$)

$$\begin{aligned} r(T(1)\,|_Q) &= \lim_{\nu\to\infty} \| T(m_\nu)\,|_Q \|^{1/m_\nu} \geqslant \lim_{\nu\to\infty} \| T(m_\nu)\,|_Q\, T(\tau_\nu)\,|_Q \|^{1/m_\nu} \\ &\geqslant \lim_{\nu\to\infty} [He^{\alpha(m_\nu+\tau_\nu)}]^{1/m_\nu} = e^\alpha. \end{aligned}$$

But this contradicts Theorem 4.1 and choice of α.

Remarks. If μ and η in (2.1) have real components, then the above decomposition is real, i.e. $E\phi$ is real whenever ϕ is real. To see this notice that $1/2\pi i \int_{C_j} (\lambda - A)^{-1}\phi(\theta)\, d\lambda$ is real for real ϕ, since it equals its complex conjugate; therefore, $E\phi$ is also real.

When P is finite dimensional, the algebraic details of the decomposition are treated in [6]. In this case, Lemma 4.3 (and thus, the adjoint theory) is unnecessary, which is fortunate since the adjoint theory of [12] is so far available only in C.

Brumley [18] studied the case when $D^0\phi = \phi(0) - \sum_{k=1}^N A_k\phi(-w_k)$ with the w_k commensurable, and obtained more precise growth estimates than those above. In particular, correcting an example due to Snow, he showed that the homogeneous equation may have unbounded solutions although all eigenvalues of A have negative real parts. In this example, there are multiple eigenvalues arbitrarily close to the imaginary axis.

Now we study the corresponding decomposition for the nonhomogeneous equation (2.1). Recall that

$$x_t(s, \phi, H) = T(t-s)\phi + K(t,s)H, \qquad t \geqslant s,$$

is the solution of (2.1) with initial value $x_s = \phi$ and forcing function H.

THEOREM 4.3. *Suppose $\alpha \notin \overline{\operatorname{Re}\sigma(A)}$ and $C = P \oplus Q$ is the decomposition provided by Theorem* 4.2. *If we use superscript P, Q to denote the component* *in that space, so $\phi = \phi^P + \phi^Q \in P \oplus Q$, then there exist functions of bounded variation $T(t)\, X_0^{P,Q}(\theta)$, functions of $t + \theta$ alone, such that*

$$x_t^P(s, \phi, H) = T(t-s)\,\phi^P + \int_s^t T(t-\xi)\, X_0^P\, dH(\xi),$$

$$x_t^Q(s, \phi, H) = T(t-s)\,\phi^Q + \int_s^t T(t-\xi)\, X_0^Q\, dH(\xi).$$

For a.e. θ in $[-r, 0]$,

$$T(t)\, X_0^P(\theta) = \lim_{j\to\infty}(1/2\pi i) \int_{C_j} e^{\lambda(t+\theta)}\, \Delta(\lambda)^{-1}\, d\lambda, \qquad (-\infty < t < \infty)$$

$$T(t)\, X_0^Q(\theta) = \lim_{j\to\infty}(1/2\pi i) \int_{\alpha-il_j}^{\alpha+il_j} e^{\lambda(t+\theta)} \Delta(\lambda)^{-1}\, d\lambda \qquad (t \geqslant 0),$$

and for any continuous H we have

$$T(t-s) \int_\sigma^s T(s-\xi)\, X_0^{P,Q}\, dH(\xi) = \int_\sigma^s T(t-\xi)\, X_0^{P,Q}\, dH(\xi)$$

if $t \geqslant s \geqslant \sigma$ (for all t, s, σ in the case of X_0^P).

Finally, there exist constants M_3, M_4 and $\epsilon > 0$ such that

$$|\, T(t)\, X_0^Q \,| \leqslant M_3 e^{(\alpha-\epsilon)t} (t \geqslant 0), \qquad \int_0^\infty |\, dT(s)\, X_0^Q \,|\, e^{-(\alpha-\epsilon)s} \leqslant M_3$$

$$|\, T(t)\, X_0^P \,| \leqslant M_4 e^{(\alpha+\epsilon)t} (t \leqslant 0), \qquad \int_{-\infty}^0 |\, dT(s)\, X_0^P \,|\, e^{-(\alpha+\epsilon)s} \leqslant M_4 .$$

Proof. It is shown in [8, Theor. 2.2] that

$$K(t, s)\, H(\theta) = -\int_s^t d_\xi X(t+\theta-\xi)[H(\xi) - H(s)] = \int_s^t X(t+\theta-\xi)\, dH(\xi)$$

where X is a matrix valued function of bounded variation on finite intervals, which vanishes on $[-r, 0)$ and has Laplace transform $\Delta(\lambda)^{-1}$.

Let $K^P(t, s)H = EK(t, s)H$, $K^Q(t, s) = (I - E)\, K(t, s)H$ for $t \geqslant s$, where E is the projection onto P along Q, and define

$$K^P(t, s)H = -T(t-s)\,|_P\, K^P(s, t)H \qquad \text{for} \quad s > t.$$

Notice that $K^{P,Q}(t, s)H = 0$ if H is constant, so as in [8, Theor. 2.2], there exist B.V. functions X^P, X^Q such that $K^{P,Q}(t, s)H = \int_s^t X^{P,Q}(t-\xi+\cdot)\, dH(\xi)$, and $X(t) = X^P(t) + X^Q(t)$ for $t \geqslant -r$. By (2.3) and the definition of K^P, for all t, s, σ we have

$$T(t-s)|_P K^P(s,\sigma)H = K^P(t,\sigma)H - K^P(t,s)H,$$

i.e. $T(t-s)|_P \int_\sigma^s X^P(s-\xi+\cdot)\,dH(\xi) = \int_\sigma^s X^P(t-\xi+\cdot)\,dH(\xi)$. The corresponding equation for X^Q holds when $t \geqslant s \geqslant \sigma$. These equations justify the notation $X^{P,Q}(t+\theta) = T(t)\,X_0^{P,Q}(\theta)$.

Let H be any continuous function which vanishes outside [0,1]; then for $t \geqslant 1$,

$$\| K^Q(t,0)H \| = \| T(t-1)|_Q K^Q(1,0)H \| \leqslant M_1 e^{(\alpha-\delta)t} \| K^Q(1,0)H \|,$$

therefore $\int_0^{1-} | d_\xi T(t-\xi)\,X_0{}^Q | \leqslant Me^{(\alpha-\delta)t}$, $t \geqslant 0$, for some constant M. If $\epsilon = (1/2)\,\delta > 0$, then

$$\begin{aligned}\int_{0+}^{\infty} | dT(\sigma)\,X_0{}^Q |\, e^{-(\alpha-\epsilon)\sigma} &= \sum_{m=1}^{\infty} e^{-(\alpha-\epsilon)m} \int_0^{1-} | d_\xi T(m-\xi)\,X_0{}^Q |\, e^{(\alpha-\epsilon)\xi} \\ &\leqslant M_3 = (Me^{-\epsilon}/1-e^{-\epsilon}) \max(1, e^\alpha) < \infty.\end{aligned}$$

To identify $T(t)\,X_0^{P,Q}$ with the contour integrals above, it suffices to observe

$$\begin{aligned}E_\mu K(t,s)H &= \operatorname*{Res}_{\lambda=\mu}(\lambda - A)^{-1} \int_s^t X(t-\xi+\cdot)\,dH(\xi) \\ &= \int_s^t \operatorname*{Res}_{\lambda=\mu}[e^{\lambda(t-\xi+\cdot)}\,\Delta(\lambda)^{-1}]\,dH(\xi)\end{aligned}$$

for every μ. It follows from the contour integral representation and the remarks above that $| T(t)\,X_0{}^Q |\, e^{-(\alpha-\epsilon)t} \to 0$ as $t \to +\infty$, and

$$| T(t)\,X_0{}^P |\, e^{-(\alpha+\epsilon)t} \to 0 \qquad \text{as} \quad t \to -\infty.$$

5. Decomposition and Estimates for Lipschitz Functions

It is sometimes convenient to use other function spaces, suited to the problem at hand. The above results and methods of proof also work, with slight modification, in the Sobolev spaces $W_p^{(1)}$ $(1 \leqslant p \leqslant \infty)$ of absolutely continuous functions on $[-r, 0]$ with p-th power integrable derivative. We will discuss the useful special case $W_\infty^{(1)}$, the Lipschitz continuous functions $\phi\colon [-r, 0] \to E^n$ with norm

$$\begin{aligned}\| \phi \|_\infty &= \max\{| \phi(0)|, \sup_{\theta_1 \neq \theta_2} | \phi(\theta_1) - \phi(\theta_2)|/| \theta_1 - \theta_2 |\} \\ &= \max\{| \phi(0)|, \operatorname{ess\,sup} | \phi' |\}.\end{aligned}$$

Let D, L be the linear functionals of (2.1) and consider the initial value problem: $x_s = \phi \in W_\infty^{(1)}$ and

$$(d/dt)\, D(x_t) = L(x_t) + h(t), \qquad \text{a.e.} \qquad t \geqslant s, \tag{5.1}$$

where $h(\cdot)$ is measurable and essentially bounded on finite intervals. There exists a unique solution $x(s, \phi, h)$ which may be written

$$x_t(s, \phi, h) = T(t - s)\phi + K(t, s)h, \qquad t \geqslant s,$$

where $\{T(t)\}_{t \geqslant 0}$ is a semigroup of bounded linear operators on $W_\infty^{(1)}$ and $K(t, s)\colon \mathscr{L}_\infty([s, t], E^n) \to W_\infty^{(1)}$ is also linear and continuous for each $t \geqslant s$. These families of operators are not strongly continuous as functions of t, but they are continuous in the weak operator topology [19, VI. 1.3], considering $W_\infty^{(1)}$ as the conjugate space of $W_1^{(1)}(E^{n*})$. Thus for any absolutely continuous $\psi\colon [-r, 0] \to E^{n*}$ (the row n-vectors) and any $\phi \in W_\infty^{(1)}$, $\langle \psi, T(t)\phi \rangle$ is continuous in $t \geqslant 0$, where

$$\langle \psi, \phi \rangle = \psi(0)\, \phi(0) + \int_{-r}^{0} \psi'(\theta)\, \phi'(\theta)\, d\theta.$$

The results used above from the theory of strongly continuous semigroups may be extended to weakly continuous semigroups with minor changes. In particular, define the weak* infinitesimal generator A by

$$A\phi = w^*\text{--}\lim_{t \to 0+} (1/t)[T(t)\phi - \phi],$$

whenever this limit exists. According to [19, IV. 13.27], $\phi = w^* - \lim_{n \to \infty} \phi_n$ if and only if $\| \phi_n \|_\infty$ is bounded and $\phi_n(\theta) \to \phi(\theta)$ uniformly in $-r \leqslant \theta \leqslant 0$. It follows easily that:

$$\mathscr{D}(A) = \{\phi \in W_\infty^{(1)} \mid \phi' \in W_\infty^{(1)} \text{ and } D(\phi') = L(\phi)\};$$
$$A\phi = \phi' \qquad \text{for} \quad \phi \in \mathscr{D}(A);$$

$\mathscr{D}(A)$ is invariant under $T(t)$ and for $\phi \in \mathscr{D}(A)$, $\psi \in W_1^{(1)}$,

$$(d/dt)\langle \psi, T(t)\phi \rangle = \langle \psi, T(t)\, A\phi \rangle = \langle \psi, AT(t)\phi \rangle;$$
$$P\sigma(T(t)) \backslash \{0\} = \exp[tP\sigma(A)]; \text{ and}$$
$$\sigma(A) = P\sigma(A) = \{\lambda \mid \det \varDelta(\lambda) = 0\}.$$

Now $\sigma(T(t))$ may be determined as in Section 3 and Theorem 4.1. Since Lemma 4.2 is known only in C, we restrict ourselves to the case when $\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geqslant \alpha\}$ is finite.

THEOREM 5.1. *If $\Lambda = \{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda \geqslant \alpha\}$ is finite, then $W_\infty^{(1)} = P \oplus Q$ where P, Q are invariant under $T(t)$, $P = \operatorname{span}\{\mathscr{M}_\lambda(A) \mid \operatorname{Re}\lambda \geqslant \alpha\}$ is finite dimensional, $\sigma(T(t)\mid_P) = \exp(t\Lambda)$ and there exist constants $M \geqslant 1$, $\delta > 0$ such that*

$$\| T(t)\phi^Q \|_\infty \leqslant M e^{(\alpha-\delta)t} \| \phi^Q \|_\infty \qquad \text{for} \quad t \geqslant 0, \quad \phi^Q \in Q.$$

Using the B.V. *functions of Theorem* 4.3, *if $x(s, \phi, h)$ is the solution of* (5.1) *with $x_s = \phi \in W_\infty^{(1)}$ and h essentially bounded on finite intervals,*

$$x_t^P(s, \phi, h) = T(t-s)\phi^P + \int_s^t T(t-\xi) X_0^P h(\xi)\, d\xi,$$

$$x_t^Q(s, \phi, h) = T(t-s)\phi^Q + \int_s^t T(t-\xi) X_0^Q h(\xi)\, d\xi$$

for $t \geqslant s$, where superscript P, Q denotes the component of x_t or ϕ in the corresponding space. We have the estimate $\| \int_s^t T(t-\xi) X_0^Q h(\xi)\, d\xi \|_\infty \leqslant$ $M_3 \operatorname{ess\,sup}_{[s,t]} \{ | h(\xi) | e^{(\alpha-\epsilon)(t-\xi)} \}$ where M_3, $\epsilon > 0$ are the constants of Theorem 4.3.

Both $T(t)\mid_P$ and $T(t) X_0^P$ have simple representations as matrix exponential functions; for details, see [6].

6. APPLICATIONS

We will apply the above machinery to a familiar task: comparing the solutions near an equilibrium point of the nonlinear equation

$$(d/dt)\{D(x_t) - G(t, x_t)\} = L(x_t) + F(t, x_t) \tag{6.1}$$

and its linear approximation

$$(d/dt)\, D(x_t) = L(x_t). \tag{6.2}$$

Here D, L are the linear functionals appearing in (2.1) and F, G are continuous functionals from $R \times C$ into E^n satisfying $F(t, 0) = 0$ and

$$| F(t, \phi) - F(t, \psi) | \leqslant \mu(\sigma) \| \phi - \psi \| \qquad \text{for} \quad \| \psi \|, \quad \| \phi \| \leqslant \sigma,$$

and the same conditions for G, where $\mu(\sigma) \to 0$ as $\sigma \to 0+$. Under these hypotheses, the initial value problem is well posed in C, and solutions may be continued as long as they remain bounded [20]. For corresponding results in $W_p^{(1)}$, $1 \leqslant p \leqslant \infty$, see [21].

THEOREM 6.1. *Suppose* $\operatorname{Re} \sigma(A) \leqslant -\gamma < 0$ *for the linear equation* (6.2), *and choose any* γ' *with* $0 < \gamma' < \gamma$. *Then there exist* $\sigma > 0$ *and* $K \geqslant 1$ *such that, if* $\phi \in C$ *and* $\| \phi \| \leqslant \sigma/K$, *then the solution* $x(s, \phi)$ *of* (6.1) *exists and satisfies*

$$\| x_t(s, \phi) \| \leqslant K \| \phi \| e^{-\gamma'(t-s)}$$

for all $t \geqslant s$.

Proof.

$$x_t(s, \phi) = T(t-s)\phi + \int_s^t T(t-\xi) X_0 \, dG(\xi, x_\xi) + \int_s^t T(t-\xi) X_0 F(\xi, x_\xi) \, d\xi$$

and $\| T(t) \| \leqslant M_1 e^{-\gamma' t}$ $(t \geqslant 0)$, $\int_0^\infty | dT(t) X_0 | e^{\gamma' t} \leqslant M_3$. Choose $\sigma > 0$ so that $2M_3\mu(\sigma) < 1$, and choose $K > M_1/(1 - 2M_3\mu(\sigma))$. If $\| \phi \| \leqslant \sigma/K$, then as long as $\| x_t \| \leqslant \sigma$,

$$\| x_t \| e^{\gamma'(t-s)} \leqslant M_1 \| \phi \| + 2\mu(\sigma) \int_s^t | dT(t-\xi) X_0 | e^{\gamma'(t-s)} \| x_\xi \|.$$

Let $u(t) = \sup\{\| x_\xi \| e^{\gamma'(\xi - s)}, s \leqslant \xi \leqslant t\}$; then

$$u(t) \leqslant M_1 \| \phi \| + 2\mu(\sigma) \int_s^t | dT(t-\xi) X_0 | e^{\gamma'(t-\xi)} u(t),$$

hence $u(t) \leqslant M_1 \| \phi \| / (1 - 2\mu(\sigma) M_3) \leqslant K \| \phi \|$, which implies the theorem.

Remark. More precise estimates are available [18] when the lags in the difference operator are commensurable.

THEOREM 6.2. (The saddle point property [3, 4]). *Suppose* $0 \notin \overline{\operatorname{Re} \sigma(A)}$, *i.e., the imaginary axis is bounded away from* $\sigma(A)$, *and make the decomposition* $C = P \oplus Q$ *with* $\alpha = 0$ *in Theorem* 4.2. *Let* E^P, E^Q *be the projections onto* P, Q *respectively. There exists* $\sigma > 0$ *such that the following holds.*

If $S = \{\phi \mid \| \phi^Q \| \leqslant \sigma/2M_1$ *and* $\| x_t(s, \phi) \| \leqslant \sigma$ *for all* $t \geqslant s\}$, *the stable manifold, then* $E^Q|_S$ *is a homeomorphism of* S *onto* $\{\phi \in Q \mid \| \phi \| \leqslant \sigma/2M_1\}$. *Further,* S *is tangent to* Q *at the origin, and any solution with initial value in* S *tends to zero as* $t \to +\infty$.

If $U = \{\phi \mid \| \phi^P \| \leqslant \sigma/2M_2$ *and* $x_t(s, \phi)$ *is a solution of* (6.1) *for all* $t \leqslant s$ *with* $\| x_t(s, \phi) \| \leqslant \sigma$, $t \leqslant s\}$, *then* $E^P|_U$ *is a homeomorphism of* U *onto* $\{\phi \in P \mid \| \phi \| \leqslant \sigma/2M_2\}$. *Further,* U *is tangent to* P *at the origin, and any solution with initial value in* U *tends to zero as* $t \to -\infty$.

The proof of Theorem 6.2 is essentially the same as in [3] and [4]. Aside from slight changes in the estimates in Theorem 4.3 compared to corresponding results in [4], the main difference in our results is that we may allow P to be infinite dimensional, i.e., we don't require the difference operator to be stable [4, 8].

An Equation with State-Dependent Lag

As a simple example of the theory of Section 5 for the Lipschitz continuous functions $W_\infty^{(1)}$, consider the equation, discussed previously by Cooke [22],

$$x'(t) = -ax(t - r(x(t))), \qquad a > 0 \tag{6.3}$$

where $r(0) = 0$, $r(x) \geqslant 0$, and we assume also

$$r(x) \leqslant r, \qquad | r(x) - r(y)| \leqslant k \mid x - y \mid.$$

We may rewrite this equation

$$\begin{aligned} x'(t) &= -ax(t) + f(x_t), \\ f(\phi) &= a\phi(0) - a\phi(-r(\phi(0))). \end{aligned}$$

Observe that $f(0) = 0$ and for ϕ, $\psi \in W_\infty^{(1)}$,

$$| f(\psi) - f(\phi)| \leqslant ak(\| \phi \|_\infty + \| \psi \|_\infty) \| \psi - \phi \|_\infty .$$

Arguing as in Theorem 6.1, it follows easily that, if $\| \phi \|_\infty$ is sufficiently small, then the solution $x(\phi)$ of (6.3) with $x_0(\phi) = \phi$ satisfies

$$\| x_t(\phi)\|_\infty \leqslant K \| \phi \|_\infty e^{-a't}, \qquad \text{where} \qquad a' = (3/4)a.$$

In fact, we will show solutions of (6.3) are asymptotic to solutions of $y'(t) = -ay(t)$, i.e., that $x(t)\, e^{at} \to C(\phi)$ as $t \to \infty$, for some $C(\phi)$ in E^n.

In Theorem 5.1, take $\alpha = -2a$ so $W_\infty^{(1)} = P \oplus Q$ where $P = \text{span}\{e^{-a\cdot}\}$, $E^P\phi(\theta) = e^{-a\theta}\phi(0)(-r \leqslant \theta \leqslant 0)$ and $T(t)\, X_0{}^P(\theta) = e^{-a(t+\theta)}$. It follows that

$$\begin{aligned} e^{2a't} \| x_t{}^Q(\phi)\|_\infty &\leqslant M_1 \| \phi^Q \|_\infty + ak \int_0^t | \, dT(t - \xi)\, X_0{}^Q \,| \, e^{2a't} \| x_\xi \|_\infty^2 \\ &\leqslant M_1 \| \phi^Q \|_\infty + akM_3K^2(\| \phi \|_\infty)^2, \end{aligned}$$

so $\| x_t{}^Q(\phi)\|_\infty = O(e^{-2a't})$ as $t \to +\infty$. Also

$$x_t{}^P(\phi)\, e^{a(t+\cdot)} = \phi(0) + \int_0^t e^{a\xi} f(x_\xi)\, d\xi \to C(\phi) = \phi(0) + \int_0^\infty e^{a\xi} f(x_\xi(\phi))\, d\xi$$

as $t \to +\infty$, since $e^{a\xi} f(x_\xi) = O(e^{a\xi} e^{-2a'\xi}) = O(e^{-(a/2)\xi})$ as $\xi \to +\infty$. Therefore as $t \to +\infty$

$$\| x_t(\phi) - C(\phi)\, e^{-a(t+\cdot)} \|_\infty = O(e^{-(3a/2)t}).$$

Remark. Cooke [23] suggested the use of the space $W_\infty^{(1)}$ in connection with retarded FDEs with asymptotically constant lag. The machinery developed above makes it possible to exploit this idea systematically, even for neutral FDEs. However, for retarded FDEs, the argument may be carried on in C, due to the "smoothing" properties of such equations: an estimate of $\sup\{|x(s)|, t - r \leqslant s \leqslant t + r\}$ yields immediately from the FDE an estimate for the derivative x' on $[t, t + r]$, i.e., an estimate of $\| x_{t+r} \|_\infty$. Thus the real power of the treatment given here is in application to neutral equations.

References

1. J. K. Hale, Linear FDEs with constant coefficients, *Contrib. Differential Equations* **2** (1963), 291–319.
2. E. Hille and R. S. Phillips, "Functional Analysis and Semigroups," Amer. Math. Soc. Colloquium Publ., Providence, R.I., Vol. 31, 1957.
3. J. K. Hale and C. Perelló, The neighborhood of a singular point of FDEs, *Contrib. Differential Equations* **3** (1964), 351–375.
4. M. A. Cruz and J. K. Hale, Exponential estimates and the saddle point property for neutral FDEs, *J. Math. Anal. Appl.*, to appear.
5. J. K. Hale, Geometric theory of FDEs, *in* "Differential Equations and Dynamical Systems" (J. K. Hale and J. P. LaSalle, Eds.), Academic Press, New York, 1967.
6. J. K. Hale and K. R. Meyer, A class of functional equations of neutral type, *Mem. Amer. Math. Soc.* **76** (1967).
7. J. K. Hale, Functional differential equations, Conference on the analytic theory of differential equations, Western Michigan Univ., 1970, Kalamazoo, Mich.
8. J. K. Hale and M. A. Cruz, Asymptotic behavior of neutral FDEs, *Arch. Rat. Mech. Anal.* **34** (1969), 331–353.
9. N. Wiener and H. R. Pitt, On absolutely convergent Fourier–Stieltjes transforms, *Duke Math. J.* **4** (1938), 420–436.
10. H. R. Pitt, A theorem on absolutely convergent trigonometrical series, *J. Math. Phys.* **16** (1937), 191–195.
11. B. W. Levinger, A folk theorem of FDEs, *J. Differential Equations* **4** (1968), 612–619.
12. D. Henry, Adjoint theory and boundary value problems for neutral linear FDEs, to appear.
13. B. Ja. Levin, "Distribution of Zeros of Entire Functions," *Translations of Math. Monographs*, v. **5**, Amer. Math. Soc., Providence, R.I., 1964.
14. R. Bellman and K. Cooke, "Differential–Difference Equations," Academic Press, New York, 1963.
15. D. Henry, Small solutions of linear autonomous FDEs, *J. Differential Equations* **8** (1970).
16. C. Cordoneanu, "Almost Periodic Functions," Interscience, New York, 1968.
17. I. C. Gohberg and M. G. Krein, "Introduction to the Theory of Linear Nonselfadjoint Operators," *Translations of Math. Monog.*, v. **18**, Amer. Math. Soc., Providence, R.I., 1969.

18. W. E. BRUMLEY, On the asymptotic behavior of solutions of differential–difference equations of neutral type, *J. Differential Equations* **7** (1970), 175–188.
19. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators," Vol. 1, Interscience, New York, 1957.
20. J. K. HALE, Forward and backward continuation for neutral FDEs, *J. Differential Equations*, to appear.
21. W. R. MELVIN, A Class of Neutral FDEs, Ph.D. thesis, Brown Univ., Providence, R.I., 1970.
22. K. L. COOKE, Asymptotic theory for the delay-differential equation $u'(t) = -au(t - r(u(t)))$, *J. Math. Anal. Appl.* **19** (1967), 160–173.
23. K. L. COOKE, Linear functional differential equations of asymptotically autonomous type, *J. Differential Equations* **7** (1970), 154–174.